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# Separability of the Toda Lattice

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**Abstract**—We study integrability of the nonperiodic, finite-dimensional Toda Lattice (TL) from the point of view of the Hamilton-Jacobi (HJ) theory of separation of variables. Known to be completely integrable in the sense of Arnol'd-Liouville, the system nonetheless cannot be integrated by the HJ method in the 'original' generalized physical position-momenta coordinates. The intrinsic (coordinate-free) characterization method by Benenti provides a tool to investigate the separability of the TL system independently of a local system of coordinates. © 2000 Elsevier Science Ltd. All rights reserved.

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## 1. INTRODUCTION

The nonperiodic, finite-dimensional Toda lattice [1] is a well-known physical model describing the dynamics of $n$ particles located on a line with an exponential interaction between neighboring atoms. In the physical position-momenta coordinates $q^i$, $p_i$, $i = 1, 2, \dots, n$, its equation of motion is given by the $2n$ equations

$$\begin{aligned} \frac{\mathrm{d}q^i}{\mathrm{d}t} &= p_i, \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} &= e^{q^{i-1}-q^i} - e^{q^i-q^{i+1}}. \end{aligned} \tag{1}$$

with the understanding that $e^{q^0-q^1} = e^{q^n-q^{n+1}} = 0$. The vector field $X_H$ of (1) is known to be Hamiltonian and completely integrable as such (see, for example [2]). The corresponding Hamiltonian function (total energy) is given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q^i-q^{i+1}} \tag{2}$$

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and the corresponding Poisson bivector $P_0$ is canonical, thus $X_H = [P_0, H]$ (throughout this paper [ , ] denotes the Schouten bracket, which generalizes the concept of the Lie bracket for two vector fields).

The Arnol'd-Liouville integrability for system (1) was first established by the Lax method [3]. Another approach to integrability of system (1) introduced in [4] is based on the existence of a second Hamiltonian representation compatible with the original one (see also [5–7]). In addition, the bi-Hamiltonian approach to integrability was employed [5] to generate hierarchies of invariant quantities of (1) based on the existence of an appropriate Fuchssteiner's master symmetry [8]. For a complete list of references and recent results in the theory of bi-Hamiltonian systems (finite and infinite dimensional), see [9].

In spite of the existing possibility of applying these two methods of integrability developed in the course of the last three decades, the question remains open if the TL system can be integrated by the classical HJ method without *a priori* knowing about its integrability. Besides, the Levi-Civita (LC) criterion (see the next section) shows that the system, unlike most of the systems of classical mechanics, *cannot* be integrated by the HJ method in the position-momenta coordinates. This result does not preclude separability in some other system of coordinates.

The theory of the intrinsic characterization of separability of Hamiltonian systems originated in the last century and has been an active area of research since then. It gives a framework for a coordinate-free characterization of the HJ separability. For a complete list of references and recent results, see [10–12], as well as the relevant references therein.

The key idea which allows one to utilize the above mentioned results is to consider the Hamiltonian system (1) in a Riemannian manifold $(M, \mathbf{g})$, where the Hamiltonian function takes the following form:

$$H = \frac{1}{2} g^{ij} p_i p_j + V(q). \tag{3}$$

Here $g^{ij}$ and $V(q)$ are the inverse of the corresponding metric tensor $\mathbf{g}$ and the potential part of the total energy, respectively. This is the subject of the considerations that follow.

## 2. SEPARABILITY IN THE ORIGINAL GENERALIZED PHYSICAL POSITION-MOMENTA VARIABLES

The first step in studying the HJ separability of a Hamiltonian system is to check if the system is separable in a given system of variables. This can be done by making use of the LC criterion [13] which gives necessary and sufficient conditions for a given Hamiltonian system to be separable in a local coordinate system. For a Hamiltonian function in the physical generalized position-momenta coordinates $q^1, \dots, q^n, p_1, \dots, p_n$, the LC criterion is given by the $1/2n(n-1)$ equations

$$\partial^i \partial^j H \partial_i H \partial_j H - \partial_i \partial^j H \partial^i H \partial_j H + \partial_i \partial_j H \partial^i H \partial^j H - \partial^i \partial_j H \partial_i H \partial^j H = 0. \tag{4}$$

Here and below $\partial_i = \frac{\partial}{\partial q^i}$, $\partial^i = \frac{\partial}{\partial p_i}$, $i \neq j$, and no summation over repeating indices is understood. Direct substitution of formula (2) into (4) shows that the Hamiltonian system (1) cannot be separated in the given coordinates. Indeed, after the substitution of (2) into (4), the left-hand side yields $(n-1)$ nowhere vanishing expressions of the form $p_i p_{i+1} e^{q^i - q^{i+1}}$, $i = 1, \dots, n-1$. We note that the three-dimensional TL, (i.e., the three particle case) was shown in [14] to be nonseparable by making use of a transformation of variables which reduced the system to a two-dimensional one which is nonseparable according to the Darboux-Bertrand Theorem system. Employing the LC criterion directly, we arrive at the same conclusion for an *arbitrary* $n$. However, this is just a local characterization.

In order to establish intrinsically that the system cannot be separated in any other system of coordinates, we shall employ the recent results by Benenti [10] giving a coordinate-free characterization of both orthogonal and nonorthogonal separation of variables.

## 3. INTRINSIC CHARACTERIZATION OF SEPARABILITY OF THE TODA LATTICE

Separability of a Hamiltonian system defined in a differential manifold $M$ means that there is a system of coordinates $\underline{q}$ (called *separable coordinates*) in which the HJ equation

$$\frac{1}{2}g^{ij}\partial_i W \partial_j W + V = h$$

corresponding to (3) has a complete solution of the form

$$W\left(\underline{q},\underline{c}\right) = W_1\left(q^1,\underline{c}\right) + \cdots + W_n\left(q^n,\underline{c}\right),$$

where $\underline{c}$ are integration constants. The $n$ first integrals in involution arising from this approach are linear or quadratic in the momenta $\underline{p}$—which corresponds to Killing vectors ($K$-vectors) and Killing 2-tensors ($K$-tensors), respectively. Benenti [10] has developed an intrinsic approach relating separability to the existence of a *single* $K$-tensor (called *the characteristic $K$-tensor*) with suitable properties (the case of orthogonal separability) or a $K$-tensor along with an Abelian Lie algebra of $K$-vectors (the case of nonorthogonal separability). This approach is a substantial improvement in comparison with the previous results, since the separation of variables in this case is based on the existence and geometrical properties of a single characteristic $K$-tensor, which corresponds to a quadratic first integral of the system under consideration (see below). We note that system (1) is defined in the Riemannian space with positive definite metric and constant curvature. Its separability is, therefore, orthogonal and can be described by the following theorem [10].

THEOREM 1. (BENENTI). *A Hamiltonian $H = G + V$ is separable in orthogonal coordinates if and only if there is a $K$-tensor $\mathbf{K}$ with pointwise simple and real eigenvalues, orthogonally integrable eigenvectors and such that* $\mathrm{d}(\mathbf{K}\cdot \mathrm{d}V) = 0$.

We shall employ the above results to study separability of the Toda lattice, a system which we have already shown to be nonseparable in the generalized physical position-momenta variables. Here and below the notation $\mathrm{d}(\mathbf{K}\cdot\mathrm{d}V) = 0$ means that the one form $\mu$, with the components $\mu_i = g_{ij}K^{jh}\partial_h V$ is closed. For brevity and to simplify the calculations, we shall consider only the cases of $n = 2$ and 3.

CASE I. $n = 2$. This case corresponds to the simplest nonperiodic Toda lattice of two particles. Separability in this case is characterized by a single $K$-tensor $\mathbf{K}$ with simple eigenvalues, since the orthogonal separability of the eigenvectors is obviously satisfied. In terms of a first integral, we need to find a quadratic first integral $F$ corresponding to $\mathbf{K}$ given by

$$F = \frac{1}{2}K^{ij}p_i p_j + U, \tag{5}$$

such that $\mathrm{d}U = \mathbf{K}\cdot\mathrm{d}V$. Following a similar treatment of the Hénon-Heiles system in [15] (see also, [16]), we can find a second quadratic first integral by finding a second Hamiltonian representation of the system in the coordinate basis $(\underline{q},\underline{p})$ defined by a Poisson bivector (different from the canonical Poisson bivector $P_0$) with constant coefficients. Indeed, the Hamiltonian vector field

$$X_H = p^1\partial_1 + p^2\partial_2 + e^{q_1-q_2}\partial^1 - e^{q_1-q_2}\partial^2 \tag{6}$$

preserves the following generic Poisson bivector $P_c$ with constant coefficients:

$$L_{X_H}(P) = 0 \Rightarrow P_c = a\partial_1\wedge\partial^1 + b\partial_1\wedge\partial^2 + b\partial_2\wedge\partial^1 + a\partial_2\wedge\partial_2.$$

Here $L_{X_H}$ denotes the operator of Lie derivation along $X_H$. A second Hamiltonian $F$ should satisfy $X_H = [P_c, F]$ from which it follows that either $a \neq 0,\ b = 0$ or $a = 0,\ b \neq 0$, otherwise, $p^1$

and $p^2$ are not linearly independent. The first case corresponds to the Hamiltonian function (2) and the canonical Poisson bivector, the second case yields, taking into account $X_H = [P_c, F]$,

$$P_c = P_2 = \partial_1 \wedge \partial^2 + \partial_2 \wedge \partial^1, \qquad F = p_1 p_2 - e^{q^1 - q^2}. \tag{7}$$

Therefore, the pair $P_2, F$ provides a second Hamiltonian representation, with a quadratic in momenta first integral. The corresponding $K$-tensor $\mathbf{K}$ has the following nonzero entries: $K^{12} = K^{21} = 1$ in the system of coordinates $\underline{q}$. The condition $\mathrm{d}(\mathbf{K} \cdot \mathrm{d}V) = 0$ is trivially satisfied: $\mathbf{K} \cdot \mathrm{d}V = \mathrm{d}V$. Although in this case there are two real eigenvalues, we can apply as well the Bertrand-Darboux-Whittaker theorem [14], according to which the existence of a second quadratic in $\underline{p}$ first integral is sufficient for separability of the corresponding system with two degrees of freedom. Therefore, we conclude that for $n = 2$, the Toda lattice is *a separable Hamiltonian system.*

CASE II. $n = 3$. In this case, in order to apply Theorem 1, we begin with the condition $\mathrm{d}(\mathbf{K} \cdot \mathrm{d}V) = 0$ on the characteristic $K$-tensor $\mathbf{K}$. Now $V(q) = e^{q^1 - q^2} + e^{q^2 - q^3}$ and $\mathbf{K}$ as a Killing tensor in an Euclidean 3-space with the standard metric whose components with respect to the coordinates $\underline{q}$ are $\delta_{ij}$, $i, j = 1, 2, 3$ is a sum of symmetrized products of the following basic Killing vectors [17,18]:

$$T_i = \partial_i, \quad (1 \le i \le 3), \qquad R_{ij} = q^i \partial_i - q^j \partial_j, \quad (1 \le i < j \le 3). \tag{8}$$

The latter imposes certain restrictions on the components of the characteristic tensor $\mathbf{K}$ as functions of $q^1, q^2, q^3$. Thus,

$$\begin{aligned} K^{11} &= a\left(q^2, q^3\right), \qquad & K^{12} = K^{21} &= b\left(\underline{q}\right), \qquad & K^{13} = K^{31} &= c\left(\underline{q}\right), \\ K^{22} &= d\left(q^1, q^3\right), \qquad & K^{23} = K^{32} &= e\left(\underline{q}\right), \qquad & K^{33} &= f\left(q^2, q^3\right). \end{aligned} \tag{9}$$

Differentiating and canceling the exponential terms, we transform the condition $\mathrm{d}(\mathbf{K} \cdot \mathrm{d}V) = 0$ into the following system of linear PDEs:

$$\begin{aligned} \partial_1 b - \partial_1 d - d - \partial_2 a + a - \partial_2 b &= 0, \\ \partial_2 c - \partial_1 b + \partial_1 e - \partial_2 b + b - c &= 0, \\ \partial_2 c - c - \partial_2 e + e - \partial_3 b + \partial_3 d &= 0, \\ \partial_2 e - \partial_2 f - f + \partial_3 d + d + \partial_3 e &= 0, \\ \partial_1 c + c - \partial_1 e - e - \partial_3 b + \partial_3 d &= 0, \\ \partial_1 e - \partial_1 f - \partial_1 d + d + \partial_3 e - e &= 0. \end{aligned} \tag{10}$$

Taking into account conditions (9), the fact that all of these functions are quadratic polynomials such that $b$ does not depend on $(q^1)^2$ and $(q^2)^2$, $c$—on $(q^1)^2$ and $(q^3)^2$ and $e$—on $(q^1)^2$ and $(q^3)^2$, we easily integrate system (10) obtaining

$$a = d = f = A = \text{const}, \qquad b = c = e = B = \text{const}. \tag{11}$$

Although, in general, not every $K$-tensor corresponds to a first integral, in our case, we get different quadratic first integrals of the system by varying the constants $A$ and $B$. Indeed, let

$$F = A \sum_{i=1}^{3} p_i^2 + 2B \sum_{\substack{i,j=1 \\ i \ne j}}^{3} p_i p_j + U(q), \tag{12}$$

where the constants $A$ and $B$ are those of (10). Let us find the constants $A$ and $B$ for which $F$ takes the form (5). In view of the conditions $L_{X_{II}}(F) = 0$ and $\{F, X_H\} = 0$, where $\{\ ,\ \}$ denotes the Poisson bracket, we arrive at the following combinations of $A$ and $B$. The case of

$A = a =$ const and $B = 0$ corresponds to the Hamiltonian function (2) (scaled by $2a$) and $\mathbf{K} = \mathbf{g}$. $A = 0$, $B = b =$ const yields the scaled first integral $2(p_1p_2 + p_2p_3 + p_1p_3) - e^{q^1-q^2} - e^{q^2-q^3}$. Thus, the most general first integral of (1), which is quadratic in momenta corresponds to the case $A = a$, $B = b$, $a, b \in \mathbb{R}$. The corresponding characteristic equation of the matrix defining $\mathbf{K}$ is

$$(A-\lambda)^3 - 3B^2(A-\lambda) + 2B^3 = 0, \tag{13}$$

where $A = a$, $B = b$. Applying Cardano's formulas to this cubic equation with respect to $A - \lambda$, we find the determinant $D$

$$D = -108\left(\frac{(2B^3)^2}{4} + \frac{(-3B^2)^3}{27}\right) = -108\left(B^6 - B^6\right) = 0,$$

for all $B \neq 0$. Therefore, all the eigenvalues of the matrix corresponding to $\mathbf{K}$ are real, moreover, for $B \neq 0$ there is one double root and one single root and if $B = 0$, there is obviously one triple root. Hence, it follows that there is no characteristic $K$-tensor $\mathbf{K}$ for system (1) satisfying the conditions of Theorem 1, which means that this Hamiltonian system in the case of $n = 3$ *is not separable in any system of coordinates.* This result entails some interesting consequences for general completely integrable Hamiltonian systems which are discussed in the next section.

## 4. CONCLUDING REMARKS

We note that Blaszak [19,20] has found sufficient conditions for separability of bi-Hamiltonian systems and described from this point of view a number of interesting physical models. This application of Benenti's characterization theorem demonstrates that these conditions are not necessary, namely there are completely integrable bi-Hamiltonian systems, which cannot be integrated by the Hamilton-Jacobi method.

On the other hand, Fernandes [21] presented an example of a completely integrable Hamiltonian system, which did not admit the bi-Hamiltonian representation based on a compatible Poisson pair. This was shown by using the availability of the action-angle coordinates for the given system. The example was the perturbed Kepler problem with the Hamiltonian function

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{1}{r} + \frac{\epsilon}{2r^2} \tag{14}$$

given in the spherical coordinates $(r, \theta, \phi)$. The Hamiltonian (14) is separable in the framework of the HJ theory. It follows from the fact that its potential is a particular case of a general potential $\tilde{V} = a(r) + b(\theta)/r^2$ admitting separation in the given coordinates. The corresponding HJ equation is given by

$$\frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 - \frac{1}{r} + \frac{1}{2mr^2}\left[\left(\frac{\partial W}{\partial \theta}\right)^2 + m\epsilon\right] + \frac{1}{2mr^2\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2 = E,$$

which after separation yields a complete integral $W$ of the form

$$W = -Et + p_\phi\phi + \int\sqrt{\beta - m\epsilon - \frac{p_\phi^2}{\sin^2}\theta}\,\mathrm{d}\theta + \int\sqrt{2m\left(E + \frac{1}{r}\right) - \frac{\beta}{r^2}}\,\mathrm{d}r$$

depending on the constants $p_\phi$, $\beta$, and $E$. Now, using the standard technique of the Hamilton-Jacobi theory, one can proceed by differentiating the last equation with respect to these constants to find the general solution to the equations of motion.

Thus, we conclude that the bi-Hamiltonian and Hamilton-Jacobi theories are not reciprocally contained in one another (at least as concerns complete integrability), but are complementary. Therefore, it is safe to assume that as far as complete integrability is concerned, there is not a universal method of integrability for finite-dimensional Hamiltonian systems.

## REFERENCES


1. M. Toda, *Theory of Nonlinear Lattices*, Springer-Verlag, New York, (1981).
2. V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaevsky, *Theory of Solitons: The Inverse Scattering Methods*, Consultants Bureau, New York, (1984).
3. H. Flashka, The Toda lattice. I. Existence of integrals, *Phys. Rev. B* **9**, 1924–1925, (1974).
4. A. Das and S. Okubo, A systematic study of the Toda lattice, *Ann. Phys.* **30**, 215–232, (1989).
5. R.L. Fernandes, On the master symmetries and bi-Hamiltonian structure of the Toda lattice, *J. of Phys. A* **26**, 3797–3803, (1993).
6. P.A. Damianou, Multiple Hamiltonian structures for Toda-type systems. Topology and physics, *J. Math. Phys.* **35**, 5511–5541, (1994).
7. R.G. Smirnov, Bi-Hamiltonian formalism: A constructive approach, *Lett. Math. Phys.* **41**, 333–347, (1997).
8. B. Fuchssteiner, Master symmetries and higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, *Prog. Theor. Phys.* **70**, 1508–1521, (1983).
9. M. Blaszak, *Multi-Hamiltonian Theory of Dynamical Systems*, Springer-Verlag, New York, (1998).
10. S. Benenti, Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation, *J. Math. Phys.* **38**, 6578–6602, (1997).
11. E.G. Kalnins, *Separation of Variables for Riemannian Spaces of Constant Curvature*, Longman Scientific & Technical, Harlow, (1980).
12. W. Miller, Jr., *Symmetry and Separation of Variables*, Addison-Wesley, New York, (1977).
13. T. Levi-Civita, Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili, *Math. Ann.* **59**, 383–397, (1904).
14. I. Marshall and S. Wojciechowski, When is a Hamiltonian system integrable?, *J. Math. Phys.* **26**, 1338–1346, (1986).
15. R.G. Smirnov, Integrability of the Hénon-Heiles system, *Appl. Math. Lett.* **11** (3), 71–74, (1998).
16. M. Blaszak and S. Rauch-Wojciechowski, A generalized Hénon-Heiles system and related integrable Newton equations, *J. Math. Phys.* **35**, 1693–1709, (1993).
17. G. Thompson, Killing tensors in spaces of constant curvature, *J. Math. Phys.* **27**, 2693–2699, (1986).
18. T. Wolf, Structural equation for Killing tensors of arbitrary rank (to appear).
19. M. Blaszak, On separability of bi-Hamiltonian chains with degenerate Poisson structures, *J. Math. Phys.* **39**, 3213–3235, (1998).
20. M. Blaszak, Bi-Hamiltonian separable chains on Riemannian manifolds, *Phys. Lett. A* **234**, 25–32, (1998).
21. R.L. Fernandes, Completely integrable bi-Hamiltonian systems, *J. Dynam. Diff. Equations* **6**, 53–69, (1994).